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Enumeration of a variant of k-noncrossing trees

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Abstract

In this paper, a variant of k-noncrossing trees where all children (labelled 1) of all internal nodes in the (l, r)-representation of the noncrossing trees must be on the left of all others is introduced and enumerated by number of nodes, root degree, labels of the eldest child or youngest child of the root, length of the leftmost path and forests. Generating functions, symbolic method, right substitutions and Lagrange-Bürmann inversion are applied to obtain the results. Known results for noncrossing trees and nondecreasing 2-noncrossing trees follow as corollaries.

Keywords: k_1-noncrossing tree, root degree, eldest child, youngest child, forest.

2020 MSC: 05C30, 05A19, 05C05.

1. Introduction

Over years, various classes of plane trees have been enumerated based on parameters such as number of nodes, leaves, degree of the root, nodes of a given degree that reside on a given level among other statistics. Recently, these results were unified by Okoth and Kasyoki in [14]. One of these classes is the set of noncrossing trees which comprises of trees drawn in the plane with nodes on the circumference of a circle and edges are straight line segments that do not intersect inside the circle, [2]. In this paper, the nodes are given numbers in counterclockwise direction around the circle with the root given number 1. The degree of a node m in a noncrossing tree is the number of edges that are incident with m. A node of degree 1 is called an endpoint. Moreover, a non-root node of degree 1 is a leaf and each node which is not a leaf is an internal node. Consider a noncrossing tree in which the edges are oriented. The number of edges that are oriented towards (respectively, away from) a node v is the indegree (respectively, outdegree) of v. An arrangement of indegree (respectively, outdegree) of nodes in the noncrossing tree is the indegree sequence (respectively, outdegree sequence) of the tree. Let (i, j) be an edge in the noncrossing tree such that i is closest to the root. If $i < j$ then the edge is an ascent. Otherwise, it is called a descent. A collection of trees is a forest.

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Noncrossing trees with n edges are one of the over 15 structures counted by the generalised Catalan number,

$$\frac{1}{2n+1}\binom{3n}{n},$$

as listed in sequence A001764 of the celebrated encyclopaedia [19]. These trees have also been enumerated by root degree [6], nodes of a given degree, maximum degree, endpoints [1], leaves, forests [2], indegree sequences and outdegree sequences [2, 15], descents [3], reachability of nodes [8, 9] among other parameters.

In 2002, Panholzer and Prodinger [18] devised a method for representing noncrossing trees as plane trees. It is called the (l, r)-representation of noncrossing trees. If (i, j) is an ascent edge in a path from the root then node j is represented by letter r. Otherwise, letter l is used. This representation has become popular in the enumeration of noncrossing trees. This representation is shown in Figure 1, where we give a noncrossing tree together with its (l, r)-representation.

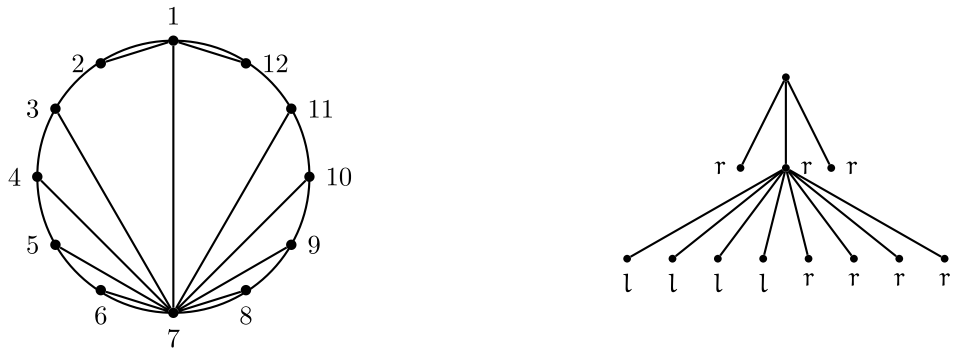

Figure 1: A 3-noncrossing tree on 12 nodes and its (l, r)-representation.

Let (x, y) be an edge in a plane tree such that x is closest to the root. Then x is said be a parent of y and y is a child of x. Children that share a parent are called siblings. The level of a node i is the number of edges in the unique path from the root to i. So, the root is at level 0. Given a level, the node on the far left is the eldest child on that level and the node at the far right is the youngest child. This means that among the siblings of the root, the one that appear on the far left is the eldest child and the one that is on the far right is the youngest child. Once we obtain the (l, r)-representation of a noncrossing tree (i.e., upon obtaining the corresponding plane tree), then the eldest and youngest child at each level can always be visualized. The path from the root that connects the eldest children is the leftmost path.

In the last-quarter century, noncrossing trees have been generalised either by considering their block graphs [4, 9, 11] or labelling the nodes to satisfy a certain condition [5, 10, 12, 13, 16, 17, 21]. Of much interest in this paper is the latter generalization, which we will now review. In 2010, Pang and Lv [17], defined a k-noncrossing tree as a noncrossing tree in which the nodes receive labels in the set $\{1, 2, \ldots, k\}$ such that if (i, j) is an ascent edge in a path from the root, then $i + j \leqslant k + 1$. Setting $k = 2$, we get 2-noncrossing trees introduced and enumerated by Yan and Liu [21]. The number of k-noncrossing trees on n nodes with root labelled s is given by

$$\frac{k-s+1}{2kn-k-s+1}\binom{(2k+1)n-k-s-1}{n-1}. \tag{1.1}$$

This result was obtained by Okoth and Wagner in [16] as a corollary of the main result which gave the formula for k-noncrossing trees in which the occurrences of nodes of given labels are specified. We get the formula for noncrossing trees on n nodes by setting $k = s = 1$ in (1.1). The set of k-noncrossing trees have also been enumerated by root degree and forests in [10]. Bijections of k-noncrossing trees with other combinatorial structures were recently constructed by Nyariaro and Okoth in [7].

We can extend the $(\mathfrak{l}, \mathfrak{r})$-representation of noncrossing trees to k-noncrossing trees. Figure 2 is a depiction of how this can be done. The subscripts are the labels of the nodes in the k-noncrossing trees.

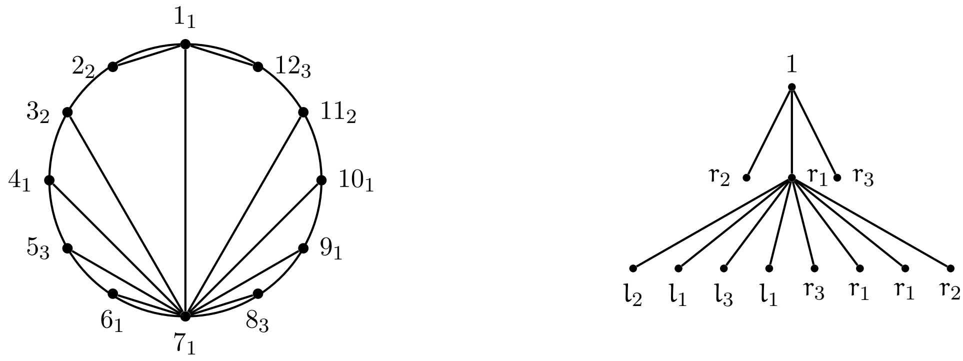

Figure 2: A 3-noncrossing tree on 12 nodes with root labelled 1 and its $(\mathfrak{l}, \mathfrak{r})$-representation.

In this work, we introduce a subclass of k-noncrossing trees which we shall call k_1-noncrossing trees.

Definition 1.1. A k_1-noncrossing tree is a k-noncrossing tree whose $(\mathfrak{l}, \mathfrak{r})$-representation is a plane tree in which for each internal node, all its children labelled $\mathfrak{l}_1$ (respectively, $\mathfrak{r}_1$) must be on the left of all children labelled $\mathfrak{l}_2, \dots, \mathfrak{l}_k$ (respectively, $\mathfrak{r}_2, \dots, \mathfrak{r}_k$).

In Figure 3 shows a 3_1-noncrossing tree on 12 nodes with root labelled 1 together with its $(\mathfrak{l}, \mathfrak{r})$-representation.

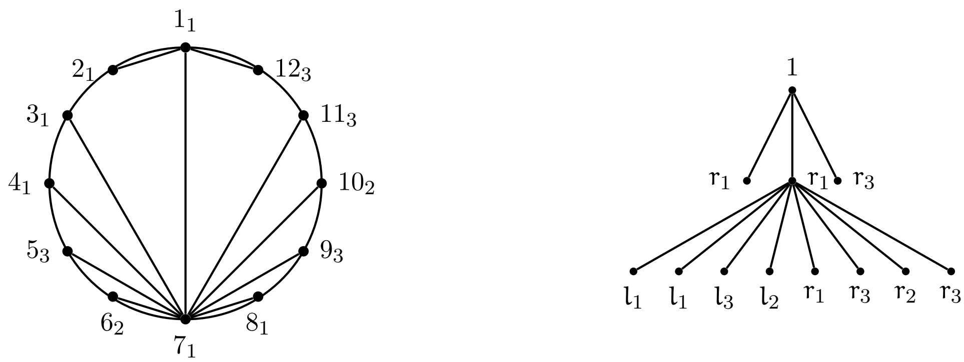

Figure 3: A 3_1-noncrossing tree on 12 nodes with root labelled 1 and its $(\mathfrak{l}, \mathfrak{r})$-representation.

We note that 1_1-noncrossing trees are just the noncrossing trees. The set of 2_1-noncrossing trees was introduced by Kariuki, Okoth and Nyamwala in [5] and therein the trees appear as nondecreasing 2-noncrossing trees. The aforementioned authors enumerated the trees by number of nodes and occurrences of nodes of label 2, root degree and forests. In the same paper, bijections between the set of 2_1-noncrossing trees and the sets of noncrossing trees, complete ternary trees and 3-Schröder paths introduced in [22] are constructed.

In [2], Flajolet and Noy introduced butterfly decomposition of noncrossing trees, a concept which has become the base for enumeration of noncrossing trees. A butterfly is an ordered pair of noncrossing trees that share a root. Let $\mathsf{N}(z)$ and $\mathsf{B}(z)$ be respectively the generating functions for noncrossing trees and butterflies, where z marks a node. Each noncrossing tree is a node together with a sequence of butterflies. The equation that relates $\mathsf{N}(z)$ and $\mathsf{B}(z)$ is thus

$$\mathsf{N}(z) = \frac{z}{1 - \mathsf{B}(z)}. \tag{1.2}$$

Since each butterfly is an ordered pair of noncrossing trees, then

$$B(z) = \frac{N(z)^2}{z}. \tag{1.3}$$

We remark that $N(z)^2$ is divided by z since when we glue together two noncrossing trees (at a node) to form a butterfly, the number of nodes reduce by one. The division is done so as to avoid over counting of nodes. We substitute (1.3) in (1.2), to get

$$N(z) = \frac{z}{1 - \frac{N(z)^2}{z}}. \tag{1.4}$$

This is the generating function for noncrossing trees. If we let $\frac{N(z)}{\sqrt{z}} = P(z)$ then (1.4) becomes

$$P(z) = \frac{\sqrt{z}}{1 - P(z)^2}.$$

The coefficient of z^n in the generating function $P(z)$ is then extracted by means of the following theorem.

Theorem 1.2 (Lagrange-Bürmann inversion, [20]). Let $g(z)$ be a generating function that satisfies the functional equation $g(z) = z\phi(g(z))$, where $\phi(0) \neq 0$. Then, $n[z^n]f(g(z)) = [s^{n-1}](f'(s)\phi(s)^n)$ where f is an arbitrary analytic function.

This paper is organized as follows. In Section 2, we obtain the generating function for k_1-noncrossing trees and the functional equation satisfied by the generating function. We then obtain the number of k_1-noncrossing trees given the number of nodes and label of the root. The study is extended to count the trees with a given root degree in Section 3 and label of the eldest or youngest child of the root in Section 4. In this work, k_1-noncrossing trees are also enumerated by the length of leftmost path. This is achieved in Section 5. In Section 6, we enumerate forests of k_1-noncrossing trees with a given number of components. For each parameter of enumeration, as corollaries, we rediscover the formulas for the corresponding noncrossing trees obtained by various authors and the formulas for the 2_1-noncrossing trees obtained by Kariuki, Okoth and Nyamwala in [5]. The paper is concluded in Section 7, wherein the authors propose ways in which this research could be extended.

2. Number of nodes

Consider the set of k_1-noncrossing trees. If one of the endpoints of an ascent edge in a k_1-noncrossing tree is labelled i then the other endpoint must have a label less than or equal to $k-i+1$. Let $N_i(z)$ be the generating function for k_1-noncrossing trees with roots labelled by i where z marks a node.
So,

$$N_i(z) = z \cdot \frac{1}{1 - \frac{N_1^2}{z}} \cdot \frac{1}{1 - \frac{N_1}{z}(N_2 + \cdots + N_{k-i+1})}. \tag{2.1}$$

We now strive to solve the system of the functional equations (2.1). Let $N_i(z) = \frac{\sqrt{zw}}{(1+w)^{i-1}}$ and $z = \frac{\sqrt{zw}(1-w)}{(1+w)^{k-1}}$. Then, from (2.1),

$$\begin{aligned} N_i(z) &= z \cdot \frac{1}{1 - \frac{zw}{z}} \cdot \frac{1}{1 - \frac{\sqrt{zw}}{z}\left(\frac{\sqrt{zw}}{1+w} + \cdots + \frac{\sqrt{zw}}{(1+w)^{k-i}}\right)} = z \cdot \frac{1}{1-w} \cdot \frac{1}{(1+w)^{i-k}} \\ &= \frac{\sqrt{zw}(1-w)}{(1+w)^{k-1}} \cdot \frac{1}{1-w} \cdot \frac{1}{(1+w)^{i-k}} = \frac{\sqrt{zw}}{(1+w)^{i-1}}. \end{aligned}$$

Since the substitutions $N_i(z) = \dfrac{\sqrt{zw}}{(1+w)^{i-1}}$ and $z = \dfrac{\sqrt{zw}(1-w)}{(1+w)^{k-1}}$ satisfy (2.1) and the latter is independent of i then they are the rights substitutions to solve the system of functional equations. We have, $w = z(1-w)^{-2}(1+w)^{2k-2}$. This equation is in a form one can apply Lagrange-Bürmann inversion to extract the coefficient of z^n in w.
Now,

$$\begin{aligned}
[z^n]N_i =&[z^n]\frac{\sqrt{zw}}{(1+w)^{i-1}} = [z^{n-1/2}]\frac{\sqrt{w}}{(1+w)^{i-1}} \\
=&\frac{1}{n-\frac{1}{2}}[w^{n-3/2}]\left(\frac{1}{2\sqrt{w}}(1+w)^{1-i} - (i-1)\sqrt{w}(1+w)^{-i}\right)(1-w)^{-2(n-1/2)} \\
&(1+w)^{2(k-1)(n-1/2)} \\
=&\frac{1}{2n-1}[w^{n-1}]\left(1+w-2(i-1)w\right)(1-w)^{-(2n-1)}(1+w)^{(k-1)(2n-1)-i}.
\end{aligned}$$

By binomial theorem, we get

$$\begin{aligned}
[z^n]N_i =& \frac{1}{2n-1}[w^{n-1}]\left(1-(2i-3)w\right)\sum_{a,b\geqslant 0}\binom{2n+a-2}{a}\binom{(k-1)(2n-1)-i}{b}w^{a+b} \\
=&\frac{1}{2n-1}\sum_{a=0}^{n-1}\binom{2n+a-2}{a}\left[\binom{(k-1)(2n-1)-i}{n-a-1} - (2i-3)\binom{(k-1)(2n-1)-i}{n-a-2}\right] \\
=&\frac{1}{2n-1}\sum_{a=0}^{n-2}\frac{(k-i)(2n-1)+2a(i-1)}{n-a-1}\binom{2n+a-2}{a}\binom{(k-1)(2n-1)-i}{n-a-2}.
\end{aligned}$$

We advertise this result as a theorem:

Theorem 2.1. There are

$$\frac{1}{2n-1}\sum_{a=0}^{n-2}\frac{(k-i)(2n-1)+2a(i-1)}{n-a-1}\binom{2n+a-2}{a}\binom{(k-1)(2n-1)-i}{n-a-2} \tag{2.2}$$

k_1-noncrossing trees with n nodes whose root is labelled i.

Setting $i = 1$ in (2.2), we find that the number of k_1-noncrossing trees with n nodes such that the root is labelled 1 is given by

$$\frac{1}{2n-1}\sum_{a=0}^{n-2}\binom{2n+a-2}{a}\binom{(k-1)(2n-1)}{n-a-1}. \tag{2.3}$$

Also, setting $i = k$ in (2.2), we obtain the following corollary.

Corollary 2.2. The number of k_1-noncrossing trees with n nodes such that the root is labelled k is

$$\frac{1}{2n-1}\sum_{a=0}^{n-1}\frac{a}{n-1}\binom{2n+a-2}{a}\binom{(k-1)(2n-2)}{n-a-1}. \tag{2.4}$$

Moreover, setting $k = 1$ in (2.3) or (2.4) $(a = n-1)$, we get that the number of 1_1-noncrossing trees (or simply noncrossing trees) is given by

$$\frac{1}{2n-1}\binom{3n-3}{n-1},$$

a result that has been obtained by several authors [19, A001764]. Also, letting $k = 2$ in (2.3) and (2.4) we rediscover

$$\frac{1}{2n-1}\sum_{a=0}^{n-2}\binom{2n+a-2}{a}\binom{2n-1}{n-a-1}$$

and

$$\frac{1}{2n-1}\sum_{a=0}^{n-1}\frac{a}{n-1}\binom{2n+a-2}{a}\binom{2n-2}{n-a-1}$$

for the number of 2_1-noncrossing trees with n nodes whose root is labelled 1 and 2 respectively which were obtained earlier by Kariuki, Okoth and Nyamwala in [5]. The following theorem gives the total number of k_1-noncrossing trees on n nodes.

Theorem 2.3. The number of k_1-noncrossing trees on n nodes is given by

$$\frac{1}{2n-1}\sum_{a=0}^{n}\binom{2n+a-2}{a}\left[\frac{3n-2a-1}{n-1}\binom{2(k-1)(n-1)}{n-a}-\frac{2n-2}{2n+a-2}\binom{(k-1)(2n-1)}{n-a}\right]. \tag{2.5}$$

Proof. The formula can be obtained by summing over all i in (2.2) and making use of telescoping and hockey-stick identity. We provide another method of proving the theorem by first summing over the respective generating functions based on the label of the root. The desired generating function is thus

$$\begin{aligned}\sum_{i=1}^{k}N_i(z) &= \sum_{i=1}^{k}\frac{\sqrt{zw}}{(1+w)^{i-1}} = \sqrt{zw}\sum_{i=1}^{k}\frac{1}{(1+w)^{i-1}} = \frac{\sqrt{zw}(1+w)}{w}\left(1-\frac{1}{(1+w)^k}\right)\\ &= \sqrt{\frac{z}{w}}(1+w-(1+w)^{-k+1}).\end{aligned}$$

So,

$$[z^n]\sum_{i=1}^{k}N_i(z) = [z^n]\sqrt{\frac{z}{w}}(1+w-(1+w)^{-k+1}) = [z^{n-1/2}]\frac{1}{\sqrt{w}}(1+w-(1+w)^{-k+1}).$$

By Lagrange-Bürmann inversion, we get

$$\begin{aligned}[z^n]\sum_{i=1}^{k}N_i(z) = &\frac{1}{n-1/2}[w^{n-3/2}]\left[\frac{1}{\sqrt{w}}(1+(k-1)(1+w)^{-k})\right.\\ &\left.-\frac{1}{2w\sqrt{w}}(1+w-(1+w)^{-k+1})\right](1-w)^{-2(n-1/2)}(1+w)^{2(k-1)(n-1/2)}\\ =&\frac{2}{2n-1}[w^{n-3/2}]\frac{1}{2w\sqrt{w}}\left[2w(k-1)(1+w)^{-k}+(1+w)^{-k+1}-(1-w)\right]\\ &(1-w)^{-(2n-1)}(1+w)^{(k-1)(2n-1)}\\ =&\frac{1}{2n-1}[w^n]\left[2w(k-1)(1-w)^{-(2n-1)}(1+w)^{(k-1)(2n-1)-k}\right.\\ &\left.+(1-w)^{-(2n-1)}(1+w)^{(k-1)(2n-1)-k+1}-(1-w)^{-2(n-1)}(1+w)^{(k-1)(2n-1)}\right]\\ =&\frac{1}{2n-1}[w^n]\left[2w(k-1)(1-w)^{-(2n-1)}(1+w)^{2(k-1)(n-1)-1}\right.\\ &\left.+(1-w)^{-(2n-1)}(1+w)^{2(k-1)(n-1)}-(1-w)^{-2(n-1)}(1+w)^{(k-1)(2n-1)}\right].\end{aligned}$$

By binomial theorem, we have

$$[z^n]\sum_{i=1}^{k} N_i(z) = \frac{1}{2n-1}\left[[w^{n-1}]2(k-1)\sum_{a,b\geqslant 0}\binom{2n+a-2}{a}\binom{2(k-1)(n-1)-1}{b}w^{a+b}\right.$$
$$+[w^n]\sum_{a,b\geqslant 0}\binom{2n+a-2}{a}\binom{2(k-1)(n-1)}{b}w^{a+b}$$
$$\left.-[w^n]\sum_{a,b\geqslant 0}\binom{2n+a-3}{a}\binom{(k-1)(2n-1)}{b}w^{a+b}\right].$$

This means that,

$$[z^n]\sum_{i=1}^{k} N_i(z) = \frac{1}{2n-1}\left[\sum_{a\geqslant 0}2(k-1)\binom{2n+a-2}{a}\binom{2(k-1)(n-1)-1}{n-a-1}\right.$$
$$\left.+\sum_{a\geqslant 0}\binom{2n+a-2}{a}\binom{2(k-1)(n-1)}{n-a} - \sum_{a\geqslant 0}\binom{2n+a-3}{a}\binom{(k-1)(2n-1)}{n-a}\right]$$
$$=\frac{1}{2n-1}\sum_{a=0}^{n}\binom{2n+a-2}{a}\left[\frac{3n-2a-1}{n-1}\binom{2(k-1)(n-1)}{n-a}\right.$$
$$\left.-\frac{2n-2}{2n+a-2}\binom{(k-1)(2n-1)}{n-a}\right].$$

This is the desired formula. □

Setting $k = 1$ in (2.5), with $n = a$, we get the total number of noncrossing trees on n nodes as

$$\frac{1}{2n-1}\binom{3n-3}{n-1}.$$

3. Root degree

In this section, we enumerate k_1-noncrossing trees by root degree and the label of the root.

Theorem 3.1. Let $\mathcal{N}_{i,j,d}$ be the set of k_1-noncrossing trees on n nodes with root labelled j and of degree d such that all the children of the root are labelled i. The number of these trees is given by

$$\frac{d}{n-1}\sum_{a=0}^{n-d-1}\frac{(2k-i-1)(n-1)+(i-1)a}{2(k-1)(n-1)-di+d}\binom{2(k-1)(n-1)-di+d}{n-a-d-1}\binom{2n+a-3}{a}. \tag{3.1}$$

Proof. The subtrees rooted at the children of the root all have their roots labelled i. Since the children of these nodes labelled i are labelled l or r, then the subtrees can be considered as k_1-noncrossing trees obtained by gluing together two k_1-noncrossing trees with roots labelled 1 and i. Thus, we extract the coefficient of

z^n in $z\left(\frac{N_1N_i}{z}\right)^d$. The division of N_1N_i by z is to avoid over counting of nodes. We have,

$$\begin{aligned}
|\mathcal{N}_{i,j,d}| =& [z^n]z\left(\frac{N_1N_i}{z}\right)^d = [z^{n+d-1}]\,(N_1N_i)^d \\
=& [z^{n+d-1}]\left(\sqrt{zw}\frac{\sqrt{zw}}{(1+w)^{i-1}}\right)^d = [z^{n-1}]w^d(1+w)^{d(1-i)} \\
=& \frac{1}{n-1}[w^{n-2}]\left(dw^{d-1}(1+w)^{d-id} + dw^d(1-i)(1+w)^{d-id-1}\right)(1+w)^{2(k-1)(n-1)} \\
& (1-w)^{-2(n-1)} \\
=& \frac{d}{n-1}[w^{n-d-1}]\left((1+w)^{2(k-1)(n-1)+d-id} + w(1-i)(1+w)^{2(k-1)(n-1)+d-id-1}\right) \\
& (1-w)^{-2(n-1)} \\
=& \frac{d}{n-1}\sum_{a\geqslant 0}\binom{2n+a-3}{a}\left[\binom{2(k-1)(n-1)+d-id}{n-d-a-1}\right. \\
& \left. -(i-1)\binom{2(k-1)(n-1)+d-id-1}{n-d-a-2}\right] \\
=& \frac{d}{n-1}\sum_{a=0}^{n-d-1}\left[\frac{(2k-i-1)(n-1)+(i-1)a}{2(k-1)(n-1)-di+d}\binom{2(k-1)(n-1)-di+d}{n-a-d-1}\right. \\
& \left.\binom{2n+a-3}{a}\right].
\end{aligned}$$

□

Since formula (3.1) is independent of j, it follows that $|\mathcal{N}_{i,e,d}| = |\mathcal{N}_{i,f,d}|$ for all e and f satisfying the coherence condition $i+e \leqslant k+1$ and $i+f \leqslant k+1$. We obtain the following result upon setting $i=1$ in (3.1) and using the fact that if a node is labelled k then its adjacent ascent node is labelled 1.

Corollary 3.2. The number of k_1-noncrossing trees on n nodes whose root is labelled k and of degree d is given by

$$\frac{d}{n-1}\sum_{a=0}^{n-d-1}\binom{2(k-1)(n-1)}{n-a-d-1}\binom{2n+a-3}{a}. \tag{3.2}$$

By setting $i=k$ in (3.1), we obtain:

Corollary 3.3. The number of k_1-noncrossing trees on n nodes with roots of degree d labelled 1 such that all children of the root are labelled k is given by

$$\frac{d}{n-1}\sum_{a=0}^{n-d-1}\frac{n+a-1}{2n-d-2}\binom{(k-1)(2n-d-2)}{n-a-d-1}\binom{2n+a-3}{a}. \tag{3.3}$$

Setting $k=1$ in (3.2) or in (3.3), we rediscover the formula for noncrossing trees on n nodes with root degree d as

$$\frac{d}{n-1}\binom{3n-d-4}{n-d-1}.$$

If $k = 2$ in (3.2) then the number of 2_1-noncrossing trees on n nodes whose root is labelled 1 and is of degree d such that all children of the root are labelled 2 is given by

$$\frac{d}{n-1}\sum_{a=0}^{n-d-1}\binom{2n-2}{n-a-d-1}\binom{2n+a-3}{a}.$$

The following theorem gives the number of k_1-noncrossing of degree d such that the labels of the children of the root are also taken into consideration. This theorem is a key result of this section as it generalizes the formula obtained in Theorem 3.1.

Theorem 3.4. There are

$$\frac{1}{n-1}\sum_{a=0}^{n-d-1}\left[\frac{(2d(k-1)-c)(n-1)+ac}{2(k-1)(n-1)-c}\binom{2(k-1)(n-1)-c}{n-a-d-1}\binom{2n+a-3}{a}\binom{d-d_1}{d_2, d_3, \ldots, d_{k-i+1}}\right] \tag{3.4}$$

k_1-noncrossing trees on n nodes whose root is labelled i and the root has d children, d_j of them are labelled j where $j = 1, 2, \ldots, k-i+1$ and $c := d_2 + 2d_3 + \cdots + (k-i)d_{k-i+1}$.

Proof. Let $N_j(z)$ be the generating function for k_1-noncrossing trees rooted at a node labelled j, where z marks a node. Since there are d_j subtrees rooted at the children of the root for $j = 1, 2, \ldots, k$, there generating function for the desired k_1-noncrossing trees in which the position of the subtrees is not taken into consideration is

$$z\left(\frac{N_1(z)^2}{z}\right)^{d_1}\left(\frac{N_1(z)N_2(z)}{z}\right)^{d_2}\cdots\left(\frac{N_1(z)N_{k-i+1}(z)}{z}\right)^{d_{k-i+1}} = z^{1-d}N_1^dN_1^{d_1}N_2^{d_2}\cdots N_{k-i+1}^{d_{k-i+1}}.$$

We extract the coefficient z^n in the generating function.

$$\begin{aligned}
[z^n]z^{1-d}N_1^dN_2^{d_2}\cdots N_{k-i+1}^{d_{k-i+1}} &= [z^{n+d-1}]N_1^dN_1^{d_1}N_2^{d_2}\cdots N_{k-i+1}^{d_{k-i+1}}\\
&= [z^{n-1+d}](\sqrt{zw})^d(\sqrt{zw})^{d_1}\cdot\left(\frac{\sqrt{zw}}{1+w}\right)^{d_2}\cdots\left(\frac{\sqrt{zw}}{(1+w)^{k-i}}\right)^{d_{k-i+1}}\\
&= [z^{n-1+d}]z^dw^d\cdot\left(\frac{1}{1+w}\right)^{d_2}\cdots\left(\frac{1}{(1+w)^{k-i}}\right)^{d_{k-i+1}}\\
&= [z^{n-1}]w^d(1+w)^{-c}
\end{aligned}$$

where $w = z(1-w)^{-2}(1+w)^{2(k-1)}$ and $c := d_2 + 2d_3 + \cdots + (k-i)d_{k-i+1}$.

By Lagrange-Bürmann inversion, we have,

$$\begin{aligned}
[z^n]z^{1-d}N_1^dN_2^{d_2}\cdots N_{k-i+1}^{d_{k-i+1}} &= \frac{1}{n-1}[w^{n-2}](dw^{d-1}(1+w)^{-c} - cw^d(1+w)^{-c-1})\\
&\quad (1-w)^{-2(n-1)}(1+w)^{2(k-1)(n-1)}\\
&= \frac{1}{n-1}(d[w^{n-d-1}](1+w)^{2(k-1)(n-1)-c} - c[w^{n-d-2}](1+w)^{2(k-1)(n-1)-c-1})\\
&\quad (1-w)^{-2(n-1)}.
\end{aligned}$$

By binomial theorem, we get

$$\begin{aligned}
&[z^n]z^{1-d}N_1^dN_2^{d_2}\cdots N_{k-i+1}^{d_{k-i+1}}\\
&= \frac{1}{n-1}\sum_{a\geqslant 0}\left[d\binom{2(k-1)(n-1)-c}{n-a-d-1} - c\binom{2(k-1)(n-1)-c-1}{n-a-d-2}\right]\binom{2n+a-3}{a}\\
&= \frac{1}{n-1}\sum_{a=0}^{n-d-1}\frac{(2d(k-1)-c)(n-1)+ac}{2(k-1)(n-1)-c}\binom{2(k-1)(n-1)-c}{n-a-d-1}\binom{2n+a-3}{a}.
\end{aligned}$$

Now, since all the children labelled 1 for each internal node are on the left of all others then there are

$$\binom{d-d_1}{d_2, d_3, \ldots, d_{k-i+1}}$$

ways of assigning labels to the children of the root so that there are d_j children labelled j for $j = 1, 2, \ldots, k-i+1$. The proof thus follows. □

We obtain equation (3.1), by setting $c = d(i-1)$ and $d_j = 0$ for all $j \neq i$ in (3.4). If $e = 0$ in Theorem 3.4 then $d_1 = d$, $d_2 = d_3 = \cdots = d_{k-i+1} = 0$ and thus, we find that there are

$$\frac{d}{n-1}\sum_{a=0}^{n-d-1}\binom{2(k-1)(n-1)}{n-a-d-1}\binom{2n+a-3}{a} \tag{3.5}$$

k_1-noncrossing trees on n nodes such that the root is labelled k and is of degree d. Equation (3.5) was also obtained in Corollary 3.2.

Setting $k = 1$ in (3.5), we find that there are

$$\frac{d}{n-1}\binom{3n-d-4}{n-d-1}$$

noncrossing trees on n nodes with root degree d.

If $k = 2$ and $i = 1$ in (3.4) then $d_1 + d_2 = d$ and $d_2 = c$. This means that $d_2 = d - d_1$ and $c = d - d_1$. We thus obtain that there are

$$\frac{1}{n-1}\sum_{a=0}^{n-d-1}\frac{(d+d_1)(n-1)+a(d-d_1)}{2n-d+d_1-2}\binom{2n-d+d_1-2}{n-a-d-1}\binom{2n+a-3}{a} \tag{3.6}$$

2_1-noncrossing trees (called nondecreasing 2-noncrossing trees in [5]) on n nodes with root labelled 1 and has d children of which d_1 are labelled 1. Summing over all d_1 and d in (3.6), we find the total number of 2_1-noncrossing trees on n nodes with root labelled 1.

If $k = 2$ and $i = 2$ in (3.4) then $d_1 = d$ and $c = 0$. It follows that there

$$\frac{d}{n-1}\sum_{a=0}^{n-d-1}\binom{2n-2}{n-a-d-1}\binom{2n+a-3}{a}$$

2_1-noncrossing trees on n nodes with root labelled 1 and has d children all labelled 2.

4. Eldest or youngest child of the root

In this section, we enumerate k_1-noncrossing trees by label of the root and label of the eldest or youngest child of the root. We prove the following result:

Theorem 4.1. The number of k_1-noncrossing trees on n nodes with root labelled i such that the eldest child of the root is labelled 1 is

$$\frac{1}{2n-1}\sum_{a=0}^{n-2}\frac{(3k-i-2)(2n-1)+2a(i-1)}{(k-1)(2n-1)-i+1}\binom{(k-1)(2n-1)-i+1}{n-a-2}\binom{2n+a-2}{a}. \tag{4.1}$$

Proof. The generating function for these trees is $N_i(z)N_1(z)^2/z$. We extract the coefficient of z^n.

$$[z^n]N_i(z)N_1(z)^2/z = [z^n]\frac{\sqrt{zw}}{(1+w)^{i-1}}\frac{(\sqrt{zw})^2}{z} = [z^{n-1/2}]\frac{w^{3/2}}{(1+w)^{i-1}}.$$

Lagrange-Bürmann inversion gives,

$$\begin{aligned}
&[z^n]N_i(z)N_1(z)^2/z \\
&= \frac{1}{n-1/2}[w^{n-3/2}]\left[\frac{\frac{3}{2}w^{1/2}}{(1+w)^{i-1}} - \frac{(i-1)w^{3/2}}{(1+w)^i}\right](1-w)^{-2(n-1/2)}(1+w)^{2(k-1)(n-1/2)} \\
&= \frac{1}{2n-1}\left[3[w^{n-2}](1+w)^{(k-1)(2n-1)-i+1} - 2(i-1)[w^{n-3}](1+w)^{(k-1)(2n-1)-i}\right] \\
&\quad (1-w)^{-(2n-1)}.
\end{aligned}$$

By binomial theorem we get,

$$\begin{aligned}
&[z^n]N_i(z)N_1(z)^2/z \\
&= \frac{1}{2n-1}\sum_{a\geqslant 0}\left[3\binom{(k-1)(2n-1)-i+1}{n-a-2} - 2(i-1)\binom{(k-1)(2n-1)-i}{n-a-3}\right]\binom{2n+a-2}{a} \\
&= \frac{1}{2n-1}\sum_{a\geqslant 0}\frac{(3k-i-2)(2n-1)+2a(i-1)}{(k-1)(2n-1)-i+1}\binom{(k-1)(2n-1)-i+1}{n-a-2}\binom{2n+a-2}{a}.
\end{aligned}$$

This completes the proof. □

Setting $i = 1$ in Theorem 4.1, we arrive at the following corollary.

Corollary 4.2. The number of k_1-noncrossing trees on n nodes with roots labelled 1 such that the eldest child of the root is also labelled 1 is

$$\frac{3}{2n-1}\sum_{a=0}^{n-2}\binom{(k-1)(2n-1)}{n-a-2}\binom{2n+a-2}{a}. \tag{4.2}$$

Setting $k = 2$ in (4.2), we obtain the formula for the number of 2_1-noncrossing trees on n nodes in which both the root and the the eldest child of the root are labelled 1 as

$$\frac{3}{2n-1}\sum_{a=0}^{n-2}\binom{2n-1}{n-a-2}\binom{2n+a-2}{a}.$$

Also, setting $k = 1$ in (4.2), we obtain the formula

$$\frac{1}{2n-1}\binom{3n-3}{n-1}$$

which enumerates noncrossing trees on n nodes.

For the remaining part of this section, we enumerate k_1-noncrossing trees in which the youngest child of the root is of a given label. If the youngest child of the root is labelled 1 then all its older siblings must be labelled 1. This has already been obtained in Corollary 2.2. So, we are interested in determining the counting formulas for k_1-noncrossing trees in which the youngest child of the root has a specified label, different from 1. We achieve this in the following theorem:

Theorem 4.3. The number of k_1-noncrossing trees on n nodes with roots labelled i such that the youngest child of the root is also labelled $j \neq 1$ is

$$\begin{aligned}
\frac{1}{2n-1}\sum_{a=0}^{n-2}\Bigg[&\frac{(3k-i-j-1)(2n-1)+2a(i+j-2)}{(k-1)(2n-1)-i-j+2}\binom{(k-1)(2n-1)-i-j+2}{n-a-2} \\
&\binom{2n+a-2}{a}\Bigg].
\end{aligned} \tag{4.3}$$

Proof. The generating function is $N_i(z)\cdot\frac{N_1(z)N_j(z)}{z}$. We proceed to extract the coefficient of z^n in the generating function.

$$[z^n]\frac{N_1(z)N_i(z)N_j(z)}{z}=[z^n]\sqrt{zw}\frac{\sqrt{zw}\sqrt{zw}}{z(1+w)^{i-1}(1+w)^{j-1}}=[z^{n-1/2}]\frac{w^{3/2}}{(1+w)^{i+j-2}}.$$

By Lagrange-Bürmann inversion, we have

$$\begin{aligned}[z^n]\frac{N_1(z)N_i(z)N_j(z)}{z}=&\frac{1}{n-1/2}[w^{n-3/2}]\left[\frac{3w^{1/2}}{2(1+w)^{i+j-2}}-(i+j-2)\frac{w^{3/2}}{(1+w)^{i+j-1}}\right]\\&(1-w)^{-2(n-1/2)}(1+w)^{2(k-1)(n-1/2)}\\=&\frac{1}{2n-1}\Big[3[w^{n-2}](1+w)^{(k-1)(2n-1)-i-j+2}\\&-2(i+j-2)[w^{n-3}](1+w)^{(k-1)(2n-1)-i-j+1}\Big](1-w)^{-(2n-1)}.\end{aligned}$$

Binomial theorem gives,

$$\begin{aligned}[z^n]\frac{N_1(z)N_i(z)N_j(z)}{z}&=\frac{1}{2n-1}\sum_{a\geqslant 0}\Bigg[\Bigg(3\binom{(k-1)(2n-1)-i-j+2}{n-a-2}\\&-2(i+j-2)\binom{(k-1)(2n-1)-i-j+1}{n-a-3}\Bigg)\binom{2n+a-2}{a}\Bigg]\\&=\frac{1}{2n-1}\sum_{a=0}^{n-2}\Bigg[\frac{(3k-i-j-1)(2n-1)+2a(i+j-2)}{(k-1)(2n-1)-i-j+2}\\&\binom{(k-1)(2n-1)-i-j+2}{n-a-2}\binom{2n+a-2}{a}\Bigg].\end{aligned}$$

□

Corollary 4.4. There are

$$\frac{1}{2n-1}\sum_{a=0}^{n-2}\frac{(3k-j-2)(2n-1)+2a(j-1)}{(k-1)(2n-1)-j+1}\binom{(k-1)(2n-1)-j+1}{n-a-2}\binom{2n+a-2}{a}\tag{4.4}$$

k_1-noncrossing trees on n nodes such that the root is labelled 1 and the youngest child of the root is labelled $j\neq 1$.

Proof. Set $i=1$ in (4.3). □

Further, setting $j=k$ in (4.4), we obtain the following result.

Corollary 4.5. The number of k_1-noncrossing trees on n nodes such that the youngest child of the root is labelled k is given by

$$\frac{1}{n-1}\sum_{a=0}^{n-2}\binom{(k-1)(2n-2)}{n-a-2}\binom{2n+a-1}{a}.$$

5. Leftmost path

In this section, we enumerate k_1-noncrossing trees by label of the root and the length of the leftmost path.

Theorem 5.1. The number of k_1-noncrossing trees on n nodes with root labelled i such that there is a leftmost path of length $\ell \geqslant 0$ and all the other nodes on the path are labelled 1 is given by

$$\frac{1}{2n-1}\sum_{a=0}^{n-\ell-1}\left[\frac{(2\ell+1)(k-1)(2n-1)-2(i-1)(n-a-1)}{(k-1)(2n-1)-i+1}\binom{2n+a-2}{a}\binom{(k-1)(2n-1)-i+1}{n-\ell-a-1}\right]. \tag{5.1}$$

Proof. We extract the coefficient of z^n in $N_i(z)\Big(\frac{N_1(z)N_1(z)}{z}\Big)^{\ell} = \frac{N_i(z)N_1(z)^{2\ell}}{z^{\ell}}$. That is,

$$[z^n]\frac{N_i(z)N_1(z)^{2\ell}}{z^{\ell}} = [z^{n+\ell}]N_i(z)N_1(z)^{2\ell}.$$

Since $N_i(z) = \frac{\sqrt{zw}}{(1+w)^{i-1}}$, where $w = z(1-w)^{-2}(1+w)^{2(k-1)}$, then

$$[z^n]\frac{N_i(z)N_1(z)^{2\ell}}{z^{\ell}} = [z^{n+\ell}]\frac{\sqrt{zw}}{(1+w)^{i-1}}\cdot(\sqrt{zw})^{2\ell} = [z^{n-1/2}]\frac{w^{\ell+1/2}}{(1+w)^{i-1}}.$$

By Lagrange-Bürmann inversion, we obtain

$$\begin{aligned}
&[z^n]\frac{N_i(z)N_1(z)^{2\ell}}{z^{\ell}}\\
&= \frac{1}{n-1/2}[w^{n-3/2}]\left((\ell+1/2)\frac{w^{\ell-1/2}}{(1+w)^{i-1}} - (i-1)\frac{w^{\ell+1/2}}{(1+w)^{i}}\right)(1-w)^{-2(n-1/2)}\\
&(1+w)^{2(k-1)(n-1/2)}\\
&= \frac{1}{2n-1}\left[(2\ell+1)[w^{n-\ell-1}](1+w)^{(k-1)(2n-1)-i+1} - 2(i-1)[w^{n-\ell-2}](1+w)^{(k-1)(2n-1)-i}\right]\\
&(1-w)^{-(2n-1)}.
\end{aligned}$$

Using binomial theorem, we get

$$\begin{aligned}
[z^n]\frac{N_i(z)N_1(z)^{2\ell}}{z^{\ell}} &= \frac{1}{2n-1}\Bigg[(2\ell+1)[w^{n-\ell-1}]\sum_{b\geqslant 0}\binom{(k-1)(2n-1)-i+1}{b}w^b\\
&\quad -2(i-1)[w^{n-\ell-2}]\sum_{b\geqslant 0}\binom{(k-1)(2n-1)-i}{b}w^b\Bigg]\sum_{a\geqslant 0}\binom{-(2n-1)}{a}w^a\\
&=\frac{1}{2n-1}\sum_{a\geqslant 0}\binom{2n+a-2}{a}\Bigg[(2\ell+1)\binom{(k-1)(2n-1)-i+1}{n-\ell-a-1}\\
&\quad -2(i-1)\binom{(k-1)(2n-1)-i}{n-\ell-a-2}\Bigg]\\
&=\frac{1}{2n-1}\sum_{a=0}^{n-\ell-1}\Bigg[\frac{(2\ell+1)(k-1)(2n-1)-2(i-1)(n-a-1)}{(k-1)(2n-1)-i+1}\binom{2n+a-2}{a}\\
&\quad \binom{(k-1)(2n-1)-i+1}{n-\ell-a-1}\Bigg].
\end{aligned}$$

This completes the proof. □

Setting $i = 1$ in (5.1), we get the following corollary.

Corollary 5.2. The number of k_1-noncrossing trees on n nodes with root labelled 1 such that there is a leftmost path of length $\ell \geqslant 0$ and all the other nodes on the path are labelled 1 is given by

$$\frac{2\ell+1}{2n-1}\sum_{a=0}^{n-\ell-1}\binom{2n+a-2}{a}\binom{(k-1)(2n-1)}{n-\ell-a-1}. \tag{5.2}$$

Further, setting $k = 1$ in (5.2), we find that the number of noncrossing trees on n nodes such that there is a leftmost path of length $\ell \geqslant 0$ is

$$\frac{2\ell+1}{2n-1}\binom{3n-\ell-3}{n-\ell-1}. \tag{5.3}$$

Also, if $\ell = 1$ in (5.1), we obtain the following result.

Corollary 5.3. The number of k_1-noncrossing trees on n nodes with root labelled i such that the eldest child of the root is labelled 1 is given by

$$\frac{1}{2n-1}\sum_{a=0}^{n-2}\frac{3(k-1)(2n-1)-2(i-1)(n-a-1)}{(k-1)(2n-1)-i+1}\binom{2n+a-2}{a}\binom{(k-1)(2n-1)-i+1}{n-a-2}. \tag{5.4}$$

Equation (5.4) was already obtained in (4.1). If we let $\ell = 0$ in (5.1), then it follows that there are

$$\frac{1}{2n-1}\sum_{a=0}^{n-1}\frac{(k-1)(2n-1)-2(i-1)(n-a-1)}{(k-1)(2n-1)-i+1}\binom{2n+a-2}{a}\binom{(k-1)(2n-1)-i+1}{n-a-1} \tag{5.5}$$

k_1-noncrossing trees on n nodes such that the root is labelled i. Setting $i = 1$ in (5.5), we find the formula for the number of k_1-noncrossing trees on n nodes such that the root is labelled 1 as

$$\frac{1}{2n-1}\sum_{a=0}^{n-1}\binom{2n+a-2}{a}\binom{(k-1)(2n-1)}{n-a-1}. \tag{5.6}$$

The formula,

$$\frac{1}{2n-1}\binom{3n-3}{n-1},$$

for the number of noncrossing trees on n nodes is rediscovered by setting $k = 1$ in (5.6). This formula is also obtained by setting $\ell = 0$ or $\ell = 1$ in (5.3).

6. Forests of k_1-noncrossing trees

In Theorem 6.1, we consider forests of k_1-noncrossing trees with the following two properties:

(i) Each component is rooted at a node with the smallest label.

(ii) The components are k_1-noncrossing trees with the root labelled k, and the components do not intersect each other.

Theorem 6.1. There are

$$\frac{1}{n-r}\binom{n}{r-1}\sum_{a=0}^{n-r-1}\binom{2(k-1)(n-r)}{a}\binom{n-2(k-2)(n-r)-1}{n-r-a-1}2^a \tag{6.1}$$

forests of k_1-noncrossing trees on n nodes with r components such that the roots of the components are labelled k.

Proof. Let $N(z)$ be the generating function for the components, i.e. k_1-noncrossing trees satisfying the given conditions, where z marks a node. We decompose the forests according to components containing node 1 as shown in Figure 4. If this component is on n nodes then there are n spaces that are to be filled by forests (possibly empty) $N_1, N_2, \dots, N_n$.

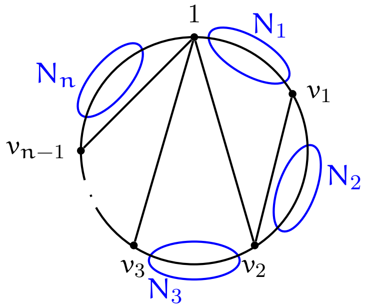

Figure 4: Decomposition of forests of k_1-noncrossing trees according to node 1.

So if $N(z) = \sum_{j\geqslant 1} n_j z^j$, then the generating function $N(z,y)$ for forests where y marks the number of components will satisfy

$$N(z,y) = 1 + y\sum_{j\geqslant 1} n_j z^j N(z,y)^j = 1 + yN(zN(z,y)).$$

Let $N(z) = \dfrac{\sqrt{zw}}{(1+w)^{k-1}}$ where $w = z(1+w)^{2k-2}(1-w)^{-2}$. Let $N(z,y) = 1+yt$. Then, $1+yt = 1+yN(z(1+yt))$ or $t = N(z(1+yt))$.

We define a as $a(s) = \frac{s}{N^{-1}(s)}$. Since $z(1+yt) = N^{-1}(t)$, then $z(1+yt) = \frac{t}{a(t)}$. Thus $t = z(1+yt)a(t)$. We apply Lagrange-Bürmann inversion to obtain

$$[z^n y^r]N(z,y) = [z^n y^{r-1}]t = \frac{1}{n}[s^{n-1}y^{r-1}]((1+ys)a(s))^n = \frac{1}{n}\binom{n}{r-1}[s^{n-r}]a(s)^n.$$

We only need to obtain $[s^{n-r}]a(s)^n$. By definition, $s = N\left(\frac{s}{a(s)}\right)$. Since $N(z) = \dfrac{\sqrt{zw}}{(1+w)^{k-1}}$, then

$$s = \frac{\sqrt{s}}{\sqrt{a(s)}} \cdot \frac{\sqrt{w\left(\frac{s}{a(s)}\right)}}{\left(1+w\left(\frac{s}{a(s)}\right)\right)^{k-1}}$$

or

$$a(s) = \frac{1}{s}\cdot\frac{w\left(\frac{s}{a(s)}\right)}{\left(1+w\left(\frac{s}{a(s)}\right)\right)^{2k-2}} \tag{6.2}$$

where $w(z)$ satisfies $w(z) = z(1+w)^{2k-2}(1-w)^{-2}$.

It follows that

$$a(s) = \frac{1}{s} \cdot \frac{s}{a(s)} \cdot \frac{\left(1+w\left(\frac{s}{a(s)}\right)\right)^{2k-2}\left(1-w\left(\frac{s}{a(s)}\right)\right)^{-2}}{\left(1+w\left(\frac{s}{a(s)}\right)\right)^{2k-2}} = \frac{1}{a(s)\left(1-w\left(\frac{s}{a(s)}\right)\right)^{2}}.$$

So,

$$a(s)^2 = \frac{1}{\left(1-w\left(\frac{s}{a(s)}\right)\right)^{2}}$$

and thus

$$a(s) = \frac{1}{\left(1-w\left(\frac{s}{a(s)}\right)\right)} \quad \text{or} \quad w\left(\frac{s}{a(s)}\right) = \frac{a(s)-1}{a(s)}.$$

Substituting $w\left(\frac{s}{a(s)}\right)$ in (6.2), we obtain

$$a(s) = \frac{1}{s} \cdot \frac{\dfrac{a(s)-1}{a(s)}}{\left(1+\dfrac{a(s)-1}{a(s)}\right)^{2k-2}} = \frac{1}{s} \cdot \frac{(a(s)-1)a(s)^{2k-3}}{(2a(s)-1)^{2k-2}}.$$

Therefore, $a(s) = 1 + s(2a(s)-1)^{2k-2}a(s)^{-(2k-4)}$. We let $b(s) = a(s) - 1$, then

$$b(s) = s(2(a(s)-1)+1)^{2k-2}a(s)^{-(2k-4)} = s\,(2b(s)+1)^{2k-2}\,(1+b(s))^{-(2k-4)}.$$

We use Lagrange-Bürmann inversion to get,

$$\begin{aligned}
[s^{n-r}]a(s)^n &= [s^{n-r}](1+b(s))^n \\
&= \frac{1}{n-r}[s^{n-r-1}]n(1+s)^{n-1}\,(2s+1)^{2(k-1)(n-r)}\,(1+s)^{-2(k-2)(n-r)} \\
&= \frac{n}{n-r}[s^{n-r-1}]\,(2s+1)^{2(k-1)(n-r)}\,(1+s)^{n-2(k-2)(n-r)-1} \\
&= \frac{n}{n-r}[s^{n-r-1}]\sum_{a,b\geqslant 0}\binom{2(k-1)(n-r)}{a}2^a\binom{n-2(k-2)(n-r)-1}{b}s^{a+b} \\
&= \frac{n}{n-r}\sum_{a=0}^{n-r-1}\binom{2(k-1)(n-r)}{a}\binom{n-2(k-2)(n-r)-1}{n-r-a-1}2^a.
\end{aligned}$$

So, the number of forests of k_1-noncrossing trees with n nodes and r components such that the components have roots labelled k is

$$\begin{aligned}
[z^n y^r]N(z,y) &= \frac{1}{n}\binom{n}{k-1}[s^{n-r}]a(s)^n \\
&= \frac{1}{n}\binom{n}{r-1}\frac{n}{n-r}\sum_{a=0}^{n-r-1}\binom{2(k-1)(n-r)}{a}\binom{n-2(k-2)(n-r)-1}{n-r-a-1}2^a. \\
&= \frac{1}{n-r}\binom{n}{r-1}\sum_{a=0}^{n-r-1}\binom{2(k-1)(n-r)}{a}\binom{n-2(k-2)(n-r)-1}{n-r-a-1}2^a.
\end{aligned}$$

□

Setting $r = 1$ in (6.1), we obtain the formula for the number of k_1-noncrossing trees with n nodes such that the root of each tree is labelled by k. The formula was already obtained in (2.2).

Also, setting $k = 2$ in (6.1), we get the number of forests of 2_1-noncrossing trees with n nodes and r components such that the root of each component is labelled by 2, i.e.,

$$\frac{1}{n-r}\binom{n}{r-1}\sum_{a=0}^{n-r-1}\binom{2n-2r}{a}\binom{n-1}{r+a}2^a.$$

This formula was obtained earlier by Kariuki, Okoth and Nyamwala in [5]. In [2], Flajolet and Noy proved that there are

$$\frac{1}{n-r}\binom{n}{r-1}\binom{3n-2r-1}{n-r-1}$$

noncrossing trees with n nodes and r components. This result is also obtained from Theorem 6.1 by setting $k = 1$.

7. Conclusion and future work

In this paper, we have introduced and enumerated a variant of k-noncrossing trees. The variant considered has all nodes labelled 1 being on the left of all other nodes. These trees are enumerated by number of nodes, root degree, label of the eldest child or youngest child of the root and forests of these trees where all components are rooted at a node with the small label and labelled by k. We propose the following ways in which the work could be extended:

(i) Enumerating other variants of k-noncrossing trees, for example, k-noncrossing trees in which are the nodes labelled 1 are on the left, followed by nodes labelled 2, and so on.

(ii) Enumerating k_1-noncrossing trees according to levels and degrees of the nodes, outdegree sequences and eldest children of the root labelled $j \neq 1$ and number of forests of these trees (not only the trees with the properties stated in Section 5).

(iii) Enumeration of k_1-noncrossing trees by the occurrences of nodes of a given labels type. This will give an equivalent result as the one obtained by Okoth and Wagner in [16].

(iv) Bijections for k_1-noncrossing trees can also be investigated.

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